

Chebyshev Series Expansion of Inverse Polynomials

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ABSTRACT. An inverse polynomial has a Chebyshev series expansion

$$1/\sum_{j=0}^k b_j T_j(x) = \sum_{n=0}^{\infty}{}' a_n T_n(x)$$

if the polynomial has no roots in $[-1, 1]$. If the inverse polynomial is decomposed into partial fractions, the a_n are linear combinations of simple functions of the polynomial roots. Also, if the first k of the coefficients a_n are known, the others become linear combinations of these with expansion coefficients derived recursively from the b_j 's. On a closely related theme, finding a polynomial with minimum *relative* error towards a given $f(x)$ is approximately equivalent to finding the b_j in $f(x)/\sum_0^k b_j T_j(x) = 1 + \sum_{k+1}^{\infty} a_n T_n(x)$, and may be handled with a Newton method providing the Chebyshev expansion of $f(x)$ is known.

1. INTRODUCTION AND SCOPE

The Chebyshev polynomials $T_n(x)$ are even or odd functions of x defined as [1, (22.3.6)][3, (3.6)]

$$(1.1) \quad T_0(x) = 1, \quad T_n(x) = \frac{n}{2} \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^m \frac{(n-m-1)!}{m!(n-2m)!} (2x)^{n-2m}, \quad n = 1, 2, 3 \dots$$

where the Gauss bracket $\lfloor \cdot \rfloor$ denotes the largest integer not greater than the number it embraces. The reverse formula is [7, p. 412][13, p. 52][14]

$$(1.2) \quad x^n = 2^{1-n} \sum_{\substack{j=0 \\ n-j \text{ even}}}^n{}' \binom{n}{(n-j)/2} T_j(x)$$

where the prime at the sum symbol means the first term (at $j = 0$ and even n) is to be halved. The polynomials are orthogonal over the interval $[-1, 1]$ with weight function $1/\sqrt{1-x^2}$ [1, (22.2.4)][7, (4.2)][2]

$$(1.3) \quad \int_{-1}^1 T_n(x) T_m(x) \frac{dx}{\sqrt{1-x^2}} = \begin{cases} \pi, & n = m = 0, \\ \pi/2, & n = m \neq 0, \\ 0, & n \neq m. \end{cases}$$

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The product rule is [1, (22.7.24)][16, (A.1)][14, (2.10)]

$$(1.4) \quad T_n(x)T_m(x) = \frac{1}{2} (T_{|m-n|}(x) + T_{m+n}(x)).$$

The indefinite integral is [7, (4.8)][13, p. 54][14, (2.12)]

$$(1.5) \quad \int T_n(x)dx = \begin{cases} T_1(x), & n = 0, \\ \frac{1}{4}T_2(x), & n = 1, \\ \frac{1}{2} \left(\frac{T_{n+1}(x)}{n+1} - \frac{T_{n-1}(x)}{n-1} \right), & n > 1, \end{cases}$$

which correlates to the derivative

$$(1.6) \quad \frac{d}{dx}T_n(x) = 2n \sum'_{\substack{l=0 \\ n-l \text{ odd}}}^{n-1} T_l(x).$$

The expansion of an inverse polynomial of degree k in a power series is [1, (3.6.16)]

$$(1.7) \quad \frac{1}{\sum_{j=0}^k d_j x^j} = \sum_{n=0}^{\infty} c_n x^n,$$

with recursively accessible [15, 0.313]

$$(1.8) \quad c_n = -\frac{1}{d_0} \sum_{\substack{j=0 \\ n-j \leq k}}^{n-1} d_{n-j} c_j, \quad c_0 = \frac{1}{d_0}, \quad n \geq 1.$$

The topic of this script is the equivalent arithmetic expansion of the inverse polynomial in a Chebyshev series,

$$(1.9) \quad \frac{1}{\sum_{j=0}^k d_j x^j} = \frac{1}{\sum_{j=0}^k b_j T_j(x)} = \sum_{n=0}^{\infty} a_n T_n(x),$$

i.e., computation of the coefficients

$$(1.10) \quad a_n = \frac{2}{\pi} \int_{-1}^1 \frac{T_n(x)}{\sum_{j=0}^k b_j T_j(x)} \frac{dx}{\sqrt{1-x^2}}$$

given the sets $\{b_j\}$ or $\{d_j\}$ that define the original function. Both sets are related via [30, (3)][31, (37)] and with (1.1) via

$$(1.11) \quad d_l = \frac{2^{l-1}}{l!} \sum_{\substack{j=0 \\ j-l \text{ even}}}^k (-)^{(j-l)/2} j^{\frac{(j+l-1)!}{2}} \frac{b_j}{\left(\frac{j-l}{2}\right)!}, \quad l = 0, \dots, k.$$

The expansion (1.9) exists if the inverse polynomial is bound in the interval $[-1, 1]$, i.e., if $\sum d_j x^j$ has no roots in $[-1, 1]$.

Characteristic generic methods of evaluating (1.10) are not reviewed here: (i) Fourier transform methods [7, (4.7)][6, 12, 10], (ii) sampling with Gauss-type quadratures [1, (25.4.38)][25, 33, 20] which effectively means using an implicit intermediate interpolatory polynomial to represent $1/\sum_{j=0}^k b_j T_j(x)$, (iii) approximation by truncation of (1.7), then insertion of (1.2), (iv) using the near-minimax properties of the Chebyshev series [23, 21].

Remark 1.1. The Fourier-Chebyshev series [18, 27][28, (3.4.1f)]

$$(1.12) \quad \frac{T_n(z) - tT_{|n-m|}(z)}{1 - 2tT_m(z) + t^2} = \sum_{k=0}^{\infty} T_{km+n}(z)t^k$$

provides special cases of polynomials with particularly simple expansions.

Remark 1.2. Insertion of $n = 1$ in (1.4) shows that the coefficients of

$$(1.13) \quad f(x) = \sum_{n=0}^{\infty} f_n T_n(x)$$

and

$$(1.14) \quad \frac{f(x)}{x} = \sum_{n=0}^{\infty} g_n T_n(x)$$

are related as

$$(1.15) \quad f_0 = g_1, \quad 2f_{n-1} = g_{n-2} + g_n, \quad n \geq 2.$$

Chapter 2 explains how the a_n of (1.10) could be computed supposed the inverse polynomial has been decomposed into partial fractions. Chapter 3 provides a recursive algorithm to derive high-indexed a_n ($n \geq k$) supposed the low-indexed a_n are given by other means. Chapter 4 touches on a (standard) integral-free method to compute approximate low-indexed a_n , and Chapter 5 deals with a specific inverse problem—which is finding the b_j from partially known a_n —related to polynomial approximants with minimum relative error.

2. THE CASE OF KNOWN PARTIAL FRACTIONS

The straight way of computing the Chebyshev series uses the decomposition of $1/\sum d_j x^j$ into partial fractions [15, 2.102], which reduces (1.9) to the calculation of the $a_{n,s}$ in

$$(2.1) \quad \frac{1}{(z-x)^s} \equiv \sum_{n=0}^{\infty} a_{n,s}(z) T_n(x),$$

where z is a root of the polynomial,

$$(2.2) \quad \sum_{j=0}^k d_j z^j = 0.$$

Sign flips of z and x in (2.1) show that

$$(2.3) \quad a_{n,s}(-z) = (-1)^{n+s} a_{n,s}(z).$$

The case of $s = 1$ has been evaluated earlier [17, (A.6)][27, 32] based on [1, (22.9.9)][35, (18)],

$$(2.4) \quad a_{n,1}(z) = \frac{2}{(z^2 - 1)^{1/2}} \frac{1}{w^n}, \quad w \equiv z + (z^2 - 1)^{1/2}, \quad z \notin [-1, 1].$$

The branch cuts of $(z^2 - 1)^{1/2}$ must be chosen such that $|w| > 1$.

Example 2.1.

$$(2.5) \quad \frac{1}{1+x^2} = \frac{i}{2} \frac{1}{i-x} - \frac{i}{2} \frac{1}{-i-x}$$

consists of two terms,

$$(2.6) \quad a_{n,1}(i) = -\frac{\sqrt{2}i^{1-n}}{(1+\sqrt{2})^n}, \quad a_{n,1}(-i) = (-)^n \frac{\sqrt{2}i^{1-n}}{(1+\sqrt{2})^n},$$

which recombine with the two factors $i/2$ and $-i/2$ to [28, (3.4.1a)]

$$(2.7) \quad \frac{1}{1+x^2} = \sqrt{2} \sum'_{n=0,2,4,6,\dots} \frac{(-)^{n/2}}{(1+\sqrt{2})^n} T_n(x).$$

Remark 2.2. The shifted Chebyshev polynomials $T^*(x) \equiv T(2x-1)$ are orthogonal over $[0, 1]$ with weight $1/\sqrt{x(1-x)}$ [1, (22.2.8)][26]. From (2.1) we get

$$(2.8) \quad \frac{1}{(z-x)^s} = 2^s \sum'_{n=0}^{\infty} a_{n,s}(2z-1) T_n^*(x),$$

and (1.5) becomes

$$(2.9) \quad \int T_n^*(x) dx = \begin{cases} \frac{1}{2} T_1^*(x), & n=0, \\ \frac{1}{8} T_2^*(x), & n=1, \\ \frac{1}{4} \left(\frac{T_{n+1}^*(x)}{n+1} - \frac{T_{n-1}^*(x)}{n-1} \right), & n>1. \end{cases}$$

Example 2.3. An example of $s=1$, $z=-1$ in (2.8) is

$$(2.10) \quad \frac{1}{1+x} = -\frac{1}{-1-x} = -2 \sum'_{n=0}^{\infty} a_{n,1}(-3) T_n^*(x),$$

where

$$(2.11) \quad a_{n,1}(-3) = \frac{(-)^{n+1}}{\sqrt{2}(3+2\sqrt{2})^n}$$

according to (2.4).

Higher second indices s of the $a_{n,s}$ are obtained from (2.1) by repeated derivation w.r.t. z ,

$$(2.12) \quad (-)^s s! \frac{1}{(z-x)^{s+1}} = \sum'_{n=0}^{\infty} \left(\frac{\partial}{\partial z} \right)^s a_{n,1}(z) T_n(x),$$

via [15, 0.432.1],

$$(2.13) \quad \begin{aligned} a_{0,s+1}(z) &= \frac{2}{s!} (-)^s \left(\frac{\partial}{\partial z} \right)^s \frac{1}{(z^2-1)^{1/2}} \\ &= 2 \sum_{l=0}^{\lfloor s/2 \rfloor} \frac{(-)^l}{l!(s-2l)!} \left(\frac{1}{2} \right)_{s-l} \frac{(2z)^{s-2l}}{(z^2-1)^{\frac{1}{2}+s-l}}, \quad s \geq 0, \end{aligned}$$

with Pochhammer's Symbol defined as [1, (6.1.22)]

$$(2.14) \quad (\alpha)_k \equiv \alpha(\alpha+1)(\alpha+2) \cdots (\alpha+k-1) = \Gamma(\alpha+k)/\Gamma(\alpha), \quad (\alpha)_0 = 1.$$

The formula

$$\begin{aligned} (s-1) \int \frac{dx}{(z-x)^s} &= \frac{1}{(z-x)^{s-1}} + A_s \\ &= (s-1) \sum_{n=0}^{\infty'} a_{n,s} \int T_n(x) dx = \sum_{n=0}^{\infty'} a_{n,s-1} T_n(x) + A_s, \quad s \geq 2 \end{aligned}$$

in conjunction with the method quoted by Cody [7, (4.8)][22, (25)] yields

$$(2.15) \quad a_{n+1,s}(z) = a_{n-1,s}(z) - \frac{2n}{s-1} a_{n,s-1}(z), \quad n \geq 1, \quad s \geq 2.$$

One needs (2.13) and

$$\begin{aligned} (2.16) \quad a_{1,s+1}(z) &= \frac{2}{\pi} \int_{-1}^1 \frac{T_1(x)}{(z-x)^{s+1}} \frac{dx}{\sqrt{1-x^2}} \\ &= -a_{0,s}(z) + z a_{0,s+1}(z) \end{aligned}$$

to start the recurrence (2.15) and to obtain all coefficients in (2.1) for a particular z . Closed form expressions for solving these recurrences in terms of Legendre Polynomials of $z/\sqrt{z^2-1} = (w^2+1)/(w^2-1)$ have been given by Elliott [8].

Remark 2.4. (2.16) may be generalized to

$$(2.17) \quad \frac{2}{\pi} \int_{-1}^1 \frac{x^l}{(z-x)^n} \frac{dx}{\sqrt{1-x^2}} = \sum_{m=0}^l (-1)^m \binom{l}{m} z^{l-m} a_{0,n-m}, \quad l < n.$$

and with (1.2) and (1.4) to

$$(2.18) \quad \frac{2}{\pi} \int_{-1}^1 \frac{x^l}{(z-x)^n} \frac{T_s(x)}{\sqrt{1-x^2}} dx = \frac{1}{2^l} \sum_{\substack{i=0 \\ l-i \text{ even}}}^l \binom{l}{\frac{l-i}{2}} [a_{|i-s|,n} + a_{i+s,n}].$$

Example 2.5. An example of degree $k=3$ is

$$(2.19) \quad \frac{1}{(4-x)^2(5+x)} = \frac{1}{78\frac{1}{2}T_0(x) - 23\frac{1}{4}T_1(x) - 1\frac{1}{2}T_2(x) + \frac{1}{4}T_3(x)}$$

$$(2.20) \quad = \frac{1}{9} \frac{1}{(4-x)^2} + \frac{1}{81} \frac{1}{(4-x)} - \frac{1}{81} \frac{1}{(-5-x)}.$$

The root at $z=4$ yields

$$(2.21) \quad a_{0,1}(4) = 2/\sqrt{15} \approx 0.5164$$

from (2.4) and

$$(2.22) \quad a_{0,2}(4) = 2 \cdot \frac{1}{2} \cdot \frac{2 \cdot 4}{\sqrt{15}^3} \approx 0.1377$$

from (2.13). The root at $z=-5$ yields

$$(2.23) \quad a_{0,1}(-5) = 2/(-\sqrt{24}) \approx -0.4082$$

from (2.4). The combined total in (2.20) is

$$(2.24) \quad a_0 = \frac{2}{\pi} \int_{-1}^1 \frac{1}{(4-x)^2(5+x)} \frac{dx}{\sqrt{1-x^2}} \approx \frac{1}{9} \cdot 0.1377 + \frac{1}{81} \cdot 0.5164 - \frac{1}{81} \cdot (-0.4082) \approx 0.0267.$$

Example 2.6. A case of $k = \infty$ is [15, 1.421.2]

$$(2.25) \quad \frac{\tanh(\pi x/2)}{x} = \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{i}{2(2m-1)} \left[\frac{1}{i(2m-1)-x} - \frac{1}{-i(2m-1)-x} \right].$$

The roots at $z = \pm i(2m-1)$ yield

$$(2.26) \quad a_{0,1}(z) = 2 / \left(\pm i \sqrt{4m^2 - 4m + 2} \right),$$

and the combined total is

$$(2.27) \quad a_0 = \frac{2}{\pi} \int_{-1}^1 \frac{\tanh(\pi x/2)}{x} \frac{dx}{\sqrt{1-x^2}} = \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{1}{(2m-1)\sqrt{m^2 - m + 1/2}} \approx 2.38.$$

Remark 2.7. From (2.4)

$$(2.28) \quad \frac{\partial a_{n,1}(z)}{\partial z} = -a_{n,1} \left[\frac{z}{z^2 - 1} + \frac{n}{(z^2 - 1)^{1/2}} \right],$$

so the (linear) propagation of the absolute relative error in the root z to the error in the coefficient $a_{n,1}$ is

$$(2.29) \quad \left| \frac{\Delta a_{n,1}}{a_{n,1}} \right| = \left| \frac{\Delta z}{z} \right| \cdot \left| \frac{z^2}{z^2 - 1} + \frac{nz}{(z^2 - 1)^{1/2}} \right|.$$

Remark 2.8. An associated factorization $\sum_{j=0}^k b_j T_j(x) \propto \prod_{m=1}^l (z_m - x)^{s_m}$, with l different roots of multiplicities s_m , decomposes the square root of the polynomial into a l -fold product of series of the prototypical forms

$$(2.30) \quad \sqrt{z - x} = \sum_{n=0}^{\infty} q_n(z) T_n(x), \quad s_m = 1,$$

$$(2.31) \quad z - x = z T_0(x) - T_1(x), \quad s_m = 2,$$

where [15, 2.576.2]

$$(2.32) \quad q_0(z) = \frac{2}{\pi} \int_0^\pi dt \sqrt{z - \cos t} = \frac{4}{\pi} \sqrt{1+z} E\left(\frac{2}{1+z}\right)$$

is related to Complete Elliptic Integrals of the Second Kind E in the notation of [1, (17.3.4)]. The $q_n(z)$ with $n \geq 1$ follow recursively using [1, (17.1.4)]. In particular, one may expand $T_n(x)$ in terms of $P_n^{(0,-1/2)}(x)$ with [11, (1.4)] to obtain

$$(2.33) \quad q_n(1) = \frac{2^{5/2}}{\pi} \sum_{l=0}^n \frac{(-n)_l (n)_l}{(3/2)_l (1/2)_l}, \quad n = 0, 1, 2, \dots$$

for the Chebyshev coefficients of $\sqrt{1-x}$. See [29] for an application.

3. RECURRENCE OF EXPANSION COEFFICIENTS

The T_n in (1.10) may be decomposed into a unique product of a polynomial by the denominator plus a remainder of polynomial degree less than k . [The dependence on x is omitted at all $T_n(x)$ for brevity.]

$$(3.1) \quad \begin{aligned} T_n &= (d_0^{(n)} T_0 + d_1^{(n)} T_1 + \dots + d_{n-k}^{(n)} T_{n-k})(b_0 T_0 + b_1 T_1 + \dots + b_k T_k) \\ &\quad + \frac{c_0^{(n)}}{2} T_0 + c_1^{(n)} T_1 + c_2^{(n)} T_2 + \dots + c_{k-1}^{(n)} T_{k-1}. \end{aligned}$$

Expansion with (1.4) yields a system of linear equations for the vector of the unknowns $d_j^{(n)}$ and $c_j^{(n)}$:

$$(3.2) \quad \left(\begin{array}{cccc|cccc} 1 & 0 & \dots & \dots & 0 & 2b_0 & b_1 & b_2 & b_3 & \dots \\ 0 & 1 & 0 & \dots & 0 & b_1 & b_0 + \frac{b_2}{2} & \frac{b_1+b_3}{2} & \frac{b_2+b_4}{2} & \dots \\ \vdots & 0 & \ddots & \ddots & \vdots & b_2 & \frac{b_1+b_3}{2} & b_0 + \frac{b_4}{2} & \frac{b_1+b_5}{2} & \dots \\ \vdots & \vdots & \ddots & 1 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & 1 & b_{k-1} & \frac{b_{k-2}+b_k}{2} & \frac{b_{k-3}}{2} & \dots & \dots \\ \hline 0 & \dots & \dots & \dots & 0 & b_k & \frac{b_{k-1}}{2} & \frac{b_{k-2}}{2} & \frac{b_{k-3}}{2} & \dots \\ \vdots & \dots & \dots & \dots & \vdots & 0 & \frac{b_k}{2} & \frac{b_{k-1}}{2} & \dots & \dots \\ \vdots & \dots & \dots & \dots & \vdots & \vdots & 0 & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \frac{b_{k-1}}{2} & \\ 0 & \dots & \dots & \dots & 0 & 0 & \dots & \dots & 0 & \frac{b_k}{2} \end{array} \right) \cdot \left(\begin{array}{c} c_0^{(n)} \\ c_1^{(n)} \\ \vdots \\ \vdots \\ c_{k-1}^{(n)} \\ d_0^{(n)} \\ d_1^{(n)} \\ \vdots \\ d_{n-k-1}^{(n)} \\ d_{n-k}^{(n)} \end{array} \right) = \left(\begin{array}{c} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{array} \right)$$

The $(n+1) \times (n+1)$ coefficient matrix $A_{r,c}$ (row index r and column index c from 0 to n) is an upper triangular matrix. It hosts a $k \times k$ unit matrix in the upper left corner, and is symmetric w.r.t. the minor diagonal that stretches from $A_{0,k}$ to $A_{n-k,n}$:

$$(3.3) \quad A_{r,c} = \delta_{r,c}, \quad 0 \leq c \leq k-1.$$

$$(3.4) \quad A_{r,k+c} = A_{c,k+r} = \begin{cases} 2b_0, & r=c=0 \\ b_c, & r=0, \quad 1 \leq c \leq k \\ \frac{1}{2}(b_{|r-c|} + b_{r+c}), & r \neq c \\ b_0 + b_{2c}/2, & r=c=1, 2, \dots, n-k \end{cases}$$

This works with the auxiliary definition

$$(3.5) \quad b_i = 0, \quad i > k \quad \text{or} \quad i < 0.$$

Insertion of (3.1) into (1.10) yields

$$(3.6) \quad a_n = 2d_0^{(n)} + \sum_{i=0}^{k-1} c_i^{(n)} a_i, \quad n \geq k,$$

which means that entire sequence a_n can be generated recursively from its first k terms, if the $d_0^{(n)}$ and $c_i^{(n)}$ are generated at the same time via (3.2) or an equivalent method. Iterated full solution of (3.2) can be avoided through recursive generation

of the set $\{d_i^{(n+1)}, c_i^{(n+1)}\}$ from $\{d_i^{(n)}, c_i^{(n)}\}$ and $\{d_i^{(n-1)}, c_i^{(n-1)}\}$ as follows:

$$(3.7) \quad d_0^{(n+1)} = d_1^{(n)} + \frac{c_{k-1}^{(n)}}{b_k} - d_0^{(n-1)},$$

$$(3.8) \quad d_1^{(n+1)} = 2d_0^{(n)} + d_2^{(n)} - d_1^{(n-1)},$$

$$(3.9) \quad d_j^{(n+1)} = d_{j-1}^{(n)} + d_{j+1}^{(n)} - d_j^{(n-1)}, \quad j = 2, 3, \dots, n-k+1.$$

$$(3.10) \quad \frac{c_0^{(n+1)}}{2} = c_1^{(n)} - \frac{b_0 c_{k-1}^{(n)}}{b_k} - \frac{c_0^{(n-1)}}{2},$$

$$(3.11) \quad c_j^{(n+1)} = c_{j-1}^{(n)} + c_{j+1}^{(n)} - \frac{b_j c_{k-1}^{(n)}}{b_k} - c_j^{(n-1)}, \quad j = 1, 2, \dots, k-1,$$

where the auxiliary definitions

$$(3.12) \quad c_j^{(n)} = 0, \quad j \geq k, \quad \text{or} \quad j < 0,$$

$$(3.13) \quad d_j^{(n)} = 0, \quad j > n-k, \quad \text{or} \quad j < 0,$$

are made to condense the notation.

Proof. Multiply (3.1) by $2T_1$ and use (1.4) as

$$(3.14) \quad \begin{aligned} 2T_1 \sum_{j=0}^{n-k} d_j^{(n)} T_j &= d_1^{(n)} T_0 + (2d_0^{(n)} + d_2^{(n)}) T_1 \\ &+ \sum_{j=2}^{n-k-1} (d_{j-1}^{(n)} + d_{j+1}^{(n)}) T_j + d_{n-k-1}^{(n)} T_{n-k} + d_{n-k}^{(n)} T_{n-k+1}, \end{aligned}$$

$$(3.15) \quad 2T_1 \sum_{j=0}^{k-1} c_j^{(n)} T_j = c_1^{(n)} T_0 + \sum_{j=1}^{k-2} (c_{j-1}^{(n)} + c_{j+1}^{(n)}) T_j + c_{k-2}^{(n)} T_{k-1} + c_{k-1}^{(n)} T_k.$$

Rewrite the last term in the previous equation

$$(3.16) \quad c_{k-1}^{(n)} T_k = \frac{c_{k-1}^{(n)}}{b_k} \sum_{j=0}^k b_j T_j - \frac{c_{k-1}^{(n)}}{b_k} b_0 T_0 - \dots - \frac{c_{k-1}^{(n)}}{b_k} b_{k-1} T_{k-1}.$$

Construct

$$\begin{aligned} 2T_1 T_n &= \left[(d_1^{(n)} + \frac{c_{k-1}^{(n)}}{b_k}) T_0 + (2d_0^{(n)} + d_2^{(n)}) T_1 \right. \\ &+ \sum_{j=2}^{n-k-1} (d_{j-1}^{(n)} + d_{j+1}^{(n)}) T_j + d_{n-k-1}^{(n)} T_{n-k} + d_{n-k}^{(n)} T_{n-k+1} \left. \right] \cdot \left[\sum_{j=0}^k b_j T_j \right] \\ &+ (c_1^{(n)} - \frac{c_{k-1}^{(n)}}{b_k} b_0) T_0 + \sum_{j=1}^{k-2} (c_{j-1}^{(n)} + c_{j+1}^{(n)} - \frac{c_{k-1}^{(n)}}{b_k} b_j) T_j + (c_{k-2}^{(n)} - \frac{c_{k-1}^{(n)}}{b_k} b_{k-1}) T_{k-1}, \end{aligned}$$

and subtract T_{n-1} for identification of the $d_j^{(n+1)}$ and $c_j^{(n+1)}$,

$$(3.17) \quad T_{n+1} = 2T_1 T_n - T_{n-1} = \left(\sum_{j=0}^{n-k+1} d_j^{(n+1)} T_j \right) \left(\sum_{j=0}^k b_j T_j \right) + \sum_{j=0}^{k-1} c_j^{(n+1)} T_j.$$

□

Example 3.1. For (2.19), we obviously have

$$(3.18) \quad c_0^{(1)} = c_2^{(1)} = c_0^{(2)} = c_1^{(2)} = 0, \quad c_1^{(1)} = c_2^{(2)} = 1.$$

in (3.1). The formulas (3.7)–(3.11) predict at $n = 2$

$$(3.19) \quad d_0^{(3)} = \frac{1}{1/4}, \quad \frac{c_0^{(3)}}{2} = -\frac{78\frac{1}{2} \cdot 1}{1/4}, \quad c_1^{(3)} = 1 - \frac{-23\frac{1}{4} \cdot 1}{1/4} - 1, \quad c_2^{(3)} = -\frac{-1\frac{1}{2} \cdot 1}{1/4}.$$

With these, (3.6) gives at $n = 3$

$$(3.20) \quad a_3 = 8 + (-314) \cdot a_0 + 93 \cdot a_1 + 6 \cdot a_2$$

which is correct since

$$(3.21) \quad a_0 \approx 0.02671606, a_1 \approx 0.00412578, a_2 \approx 0.00087916, a_3 \approx 0.00013030.$$

The next step of the recursion is

$$(3.22) \quad d_0^{(4)} = \frac{6}{1/4}, \quad \frac{c_0^{(4)}}{2} = 93 - \frac{78\frac{1}{2} \cdot 6}{1/4}, \quad c_1^{(4)} = 2 \cdot (-314) + 6 - \frac{-23\frac{1}{4} \cdot 6}{1/4}, \quad c_2^{(4)} = 93 - \frac{-1\frac{1}{2} \cdot 6}{1/4} - 1,$$

$$(3.23) \quad a_4 = 48 + (-1791) \cdot a_0 + (-64) \cdot a_1 + 128 \cdot a_2$$

which is also correct with

$$(3.24) \quad a_4 \approx 0.00002159.$$

4. APPROXIMATION BY THE TRUNCATED CHEBYSHEV SERIES

Approximations \hat{a}_n to the a_n of (1.9) may be calculated assuming that the a_n are negligible beyond some index N :

$$(4.1) \quad \frac{1}{\sum_{j=0}^k b_j T_j(x)} \approx \sum_{n=0}^N \hat{a}_n T_n(x).$$

If this equation is multiplied by $2 \sum b_j T_j$, and we stay with (3.5) to keep the notation simple,

$$(4.2) \quad 2 \approx \sum_{n=0}^N \hat{a}_n \sum_{l=0}^{k+n} (b_{n-l} + b_{l+n} + b_{l-n}) T_l.$$

If the coefficients in front of T_0 to T_N are set equal on both sides, a system of linear equations for the \hat{a}_n ensues:

$$(4.3) \quad \begin{pmatrix} b_0 & b_1 & b_2 & b_3 & \dots \\ b_1 & 2b_0 + b_2 & b_1 + b_3 & b_2 + b_4 & \dots \\ b_2 & b_1 + b_3 & 2b_0 + b_4 & b_1 + b_5 & \dots \\ b_3 & b_2 + b_4 & b_1 + b_5 & 2b_0 + b_6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} \hat{a}_0 \\ \hat{a}_1 \\ \vdots \\ \vdots \\ \hat{a}_N \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

where the coefficient matrix $B_{r,c}$ is symmetric and has a band width of $2k + 1$:

$$(4.4) \quad B_{r,c} = \begin{cases} b_c, & r = 0 \\ b_r, & c = 0 \\ 2b_0 + b_{2r}, & r = c \neq 0 \\ b_{|r-c|} + b_{r+c}, & r \neq c, \quad c > 0, \quad r > 0 \end{cases}$$

This gives access to a set of approximate, low-indexed a_n with no need to evaluate integrals nor reference to the roots of $\sum b_j T_j$.

Example 4.1. Again for (2.19), the choice of $N = 3$ yields the coefficient vector

$$(4.5) \quad \hat{a}_0 = 0.02671602, \hat{a}_1 = 0.00412567, \hat{a}_2 = 0.00087845, \hat{a}_3 = 0.00012696.$$

At $N = 4$, this improves to

$$(4.6) \quad \begin{aligned} \hat{a}_0 &= 0.02671606, \hat{a}_1 = 0.00412578, \hat{a}_2 = 0.00087914, \hat{a}_3 = 0.00013019, \\ \hat{a}_4 &= 0.00002111, \end{aligned}$$

which is close to the exact results in (3.21) and (3.24). At $N = 5$, this improves further to

$$(4.7) \quad \begin{aligned} \hat{a}_0 &= 0.02671606, \hat{a}_1 = 0.00412578, \hat{a}_2 = 0.00087916, \hat{a}_3 = 0.00013029, \\ \hat{a}_4 &= 0.00002158, \dots \end{aligned}$$

Remark 4.2. The matrix in (4.3) is the approximate, square upper left $(N + 1) \times (N + 1)$ submatrix of the “exact” solution. The approximate solution obtained could be considered as if the terms $\sum_{n=N+1}^{l+k} \hat{a}_n (b_{n-l} + b_{n+l} + b_{l-n})$ of (4.2) in the l ’th row of the system of linear equations had been neglected (as if the columns $N + 1$ up to $l + k$ had been chopped off). The neglected sum is nonzero only if $l \geq N + 1 - k$. An idea of an improvement of this algorithm is: reduce the \hat{a}_n ($N + 1 \leq n \leq l + k$) in the neglected terms via (3.6) to a linear combination of $\hat{a}_{1,\dots,k}$, and re-introduce (add) these components (in)to the matrix—add the constant to the right hand side—in these rows $l \geq N + 1 - k$. This update of the system of linear equations reduces the rank of the matrix and does therefore not improve on what is obtained from (4.3).

The algorithm may be extended to the division problem of finding the \hat{a}_n from given f_n in

$$(4.8) \quad \frac{f(x)}{\sum_{j=0}^k b_j T_j(x)} \approx \sum_{n=0}^N \hat{a}_n T_n(x); \quad f(x) \equiv \sum_{n=0}^{\infty} f_n T_n(x),$$

with the right hand side in (4.3) replaced as follows:

$$(4.9) \quad \sum_{c=0}^N B_{r,c} \hat{a}_c = \begin{cases} f_r, & r = 0, \\ 2f_r, & r = 1, 2, 3, \dots \end{cases}$$

Example 4.3. The Chebyshev series of $f(x) = \sin(\frac{\pi}{2}x)/x$ starts with [5, 30]

$$(4.10) \quad f_0 = \pi \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)(s!)^2} \left(\frac{\pi}{4}\right)^{2s} \approx 2.552557924804531760415274,$$

$$(4.11) \quad f_2 \approx -0.2852615691810360095702941,$$

$$(4.12) \quad f_4 \approx 0.009118016006651802497767923,$$

$$(4.13) \quad f_6 \approx -0.0001365875135419666724364765,$$

$$(4.14) \quad f_8 \approx 0.000001184961857661690108290062,$$

$$(4.15) \quad f_n = \begin{cases} 4(-)^{n/2} \sum_{s=n/2}^{\infty} J_{2s+1}(\pi/2), & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

(Schonfelder [30] lists $2f_{2n}/\pi$ for $n \leq 16$.) If we approximate $f(x)$ by the polynomial $\sum_{n=0}^4 f_n T_n(x)$, calculation of the \hat{a}_n in

$$(4.16) \quad \frac{f(x)}{\sum_{j=0}^4 f_j T_j(x)} \approx \sum_{n=0}^N \hat{a}_n T_n(x)$$

via (4.9) at $N = 8$ predicts the relative error

$$(4.17) \quad \begin{aligned} \sum_{n=0}^N \hat{a}_n T_n(x) - 1 &\approx -6.74 \cdot 10^{-8} T_0(x) - 9.97 \cdot 10^{-7} T_2(x) \\ &\quad -1.23 \cdot 10^{-5} T_4(x) - 1.09 \cdot 10^{-4} T_6(x) - 1.13 \cdot 10^{-5} T_8(x). \end{aligned}$$

Remark 4.4. The functional relations (4.8) hold also for the shifted Chebyshev polynomials $T^*(x)$:

$$(4.18) \quad \frac{f(x)}{\sum_{j=0}^k b_j T_j^*(x)} \approx \sum_{n=0}^N \hat{a}_n T_n^*(x); \quad f(x) \equiv \sum_{n=0}^{\infty} f_n T_n^*(x),$$

5. CHEBYSHEV APPROXIMATION FOR THE RELATIVE ERROR

The previous example of a truncated Chebyshev series had a maximum absolute error estimated at $\sum_{n=6}^8 |f_n| \approx 0.000138$ if terms up to $k = 4$ were retained, and the maximum relative error of the same polynomial was estimated at $\sum_{n=0}^8 |\hat{a}_n| - 1 \approx 0.000134$ —dominated by the \hat{a}_6 term in (4.17). To optimize the approximation of $f(x)$ for the relative error in $[-1, 1]$, one would rather like to find the $k + 1$ coefficients b_j in (4.8) which force the relative error to be close to zero in the sense of

$$(5.1) \quad \hat{a}_0 = 2, \quad \hat{a}_1 = \hat{a}_2 = \hat{a}_3 = \dots = \hat{a}_k = 0.$$

As an inversion of the problem of Sec. 4, the matrix B in (4.9) is presumed unknown (up to some symmetry), and the first $k + 1$ elements of the vector \hat{a}_c and all elements of f_r are known. The rationale is that removal of the ripples of $T_1(x)$ to $T_k(x)$ from the quotient expansion leaves a quotient with an appropriate number of “critical” points required by the alternating maximum theorem [7, 24, 36].

Remark 5.1. The case $r = 0$ in (4.9) in conjunction with (5.1) mandate

$$(5.2) \quad b_0 = f_0/2.$$

Finding the constituents b_j of B that solve the bi-linear equation (4.9) may proceed with a vectorized first-order Newton method as follows:

- Chose a start solution b_j , for example the obvious

$$(5.3) \quad b_j = \begin{cases} f_0/2, & j = 0 \\ f_j, & j = 1, 2, \dots, k \end{cases}$$

- Compute the \hat{a}_n ($n = 0, \dots, N$) from b_j by solving the linear system of equations (4.9).

- Compute an approximate $(N + 1) \times k$ Jacobi matrix

$$(5.4) \quad J_{r,c} = \begin{pmatrix} \frac{\partial \hat{a}_0}{\partial b_1} & \frac{\partial \hat{a}_0}{\partial b_2} & \cdots & \frac{\partial \hat{a}_0}{\partial b_k} \\ \frac{\partial \hat{a}_1}{\partial b_1} & \frac{\partial \hat{a}_1}{\partial b_2} & \cdots & \frac{\partial \hat{a}_1}{\partial b_k} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial \hat{a}_N}{\partial b_1} & \frac{\partial \hat{a}_N}{\partial b_2} & \cdots & \frac{\partial \hat{a}_N}{\partial b_k} \end{pmatrix}$$

by partial derivation of the first $N + 1$ equations of (4.9) w.r.t. the b_j , i.e., by solving the k systems of $N + 1$ linear equations

$$(5.5) \quad \sum_{c=0}^N B_{r,c} J_{c,j} = - \begin{pmatrix} \hat{a}_1 & \hat{a}_2 & \hat{a}_3 & \cdots & \hat{a}_{k-1} & \hat{a}_k \\ \hat{a}_0 + \hat{a}_2 & \hat{a}_1 + \hat{a}_3 & \hat{a}_2 + \hat{a}_4 & \cdots & \cdots & \hat{a}_{k-1} + \hat{a}_{k+1} \\ \hat{a}_1 + \hat{a}_3 & \hat{a}_0 + \hat{a}_4 & \hat{a}_1 + \hat{a}_5 & \cdots & \cdots & \hat{a}_{k-2} + \hat{a}_{k+2} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \hat{a}_{N-1} & \hat{a}_{N-2} & \hat{a}_{N-3} & \cdots & \hat{a}_{N-k-1} & \hat{a}_{N-k} \end{pmatrix}$$

for $r = 0, \dots, N$ and $j = 0, \dots, k - 1$. The column $\partial \hat{a}_c / \partial b_0$ of the Jacobi matrix is not calculated, as b_0 is assumed fixed according to (5.2).

- Compute the next iterated solution $b_j + \Delta_j$ ($j = 1, 2, \dots, k$) of the polynomial coefficients by solving the system of k linear equations

$$(5.6) \quad \sum_{j=1}^k \frac{\partial \hat{a}_l}{\partial b_j} \Delta_j = -\hat{a}_l, \quad l = 1, \dots, k$$

for the first-order differences Δ_j . This equation is the first-order Taylor expansion of \hat{a}_l as a function of the b_j set to the target (5.1) for this update. The $k \times k$ coefficient matrix $\partial \hat{a}_l / \partial b_j$ is a square submatrix of the Jacobi matrix calculated in the previous step.

- Return to the second bullet for the next cycle until the \hat{a}_0 to \hat{a}_k are sufficiently close to (5.1).

Remark 5.2. This algorithm involves only f_0 to f_N , but no higher order approximants to $f(x)$. It therefore adapts a polynomial of degree k to a polynomial of degree N .

Example 5.3. The error terms (4.17) for the polynomial $\sum_{j=0}^4 b_j T_j(x)$ change to

$$(5.7) \quad \begin{aligned} \sum_{n=0}^N \hat{a}_n T_n(x) - 1 &\approx 5.2 \cdot 10^{-12} T_0(x) + 4.7 \cdot 10^{-11} T_2(x) + 6.3 \cdot 10^{-12} T_4(x) \\ &\quad - 1.08 \cdot 10^{-4} T_6(x) - 1.11 \cdot 10^{-5} T_8(x) \end{aligned}$$

after one Newton iteration, reducing the relative error to $\sum_{n=0}^8 |\hat{a}_n| - 1 \approx 0.000119$. During further iteration cycles the relative error stays about the same because it is dominated by $\hat{a}_6 T_6(x)$ which is out of reach of the polynomial base with $k = 4$.

Example 5.4. An IEEE “single” precision accuracy of $f(x) = \sin(\frac{\pi}{2}x)/x$ with a relative error smaller than $2^{-24} \approx 6.0 \cdot 10^{-8}$ needs $k = 8$. Truncation of the Chebyshev series for $f(x)$ after $k = 8$ yields an estimated maximum absolute error of $\sum_{n=k+1}^N |f_n| \approx 6.7 \cdot 10^{-9}$ evaluated at $N = 16$. The relative error of the same polynomial is also $\sum_{n=0}^N |\hat{a}_n| - 1 \approx 6.7 \cdot 10^{-9}$. After four Newton iterations, this

value drops to $5.9 \cdot 10^{-9}$ with coefficients given in the following table—remaining very close to those cited after (4.10):

n	b_n
0	1.276278962402265880207637
2	-0.2852615691810328617761446
4	$0.9118016006289075331306166 \cdot 10^{-2}$
6	$-0.1365874893444115901818408 \cdot 10^{-3}$
8	$0.1184206224108742454613850 \cdot 10^{-5}$

Example 5.5. $g(x) = \cos(\frac{\pi}{2}x)$ has the expansion coefficients [5, 23, 30]

$$(5.8) \quad g_n = \begin{cases} 2(-)^{n/2} J_n(\pi/2), & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

The approximation $g(x) \approx \sum_{n=0}^k g_n T_n(x)$ has an estimated maximum absolute error of $\sum_{n=k+1}^N |g_n| \approx 4.7 \cdot 10^{-8}$ for the polynomial of degree $k = 8$ evaluated at $N = 16$. Because $g(x)$ is zero at both ends of the interval $[-1, 1]$, the algorithm does not find polynomials $\sum_{j=0}^k b_j T_j(x)$ with a uniformly convergent Chebyshev expansion of the relative error—any \hat{a}_n obtained depend strongly on N . We therefore “lift” both zeros by looking at $f(x) = \cos(\frac{\pi}{2}x)/(1-x^2)$ instead, which has the expansion coefficients

$$(5.9) \quad f_0 = \pi J_1(\pi/2),$$

$$(5.10) \quad f_2 = f_0 - 2g_0,$$

$$(5.11) \quad f_n = 2f_{n-2} - f_{n-4} - 4g_{n-2}, \quad n = 4, 6, 8, \dots,$$

$$(5.12) \quad f_n = 0, \quad n \text{ odd}.$$

Truncation of the Chebyshev series for $f(x)$ after $k = 4$ yields an estimated maximum absolute error of $\sum_{n=k+1}^N |f_n| \approx 2.7 \cdot 10^{-5}$ evaluated at $N = 8$. The relative error of the same polynomial is $\sum_{n=0}^N |\hat{a}_n| - 1 \approx 3.3 \cdot 10^{-5}$. After four Newton iterations, this value drops to $3.1 \cdot 10^{-5}$ with coefficients given in the following table:

n	b_n
0	0.8903651967922106931461297
2	-0.1072744347398521266520654
4	0.002332103968386755210894198

Example 5.6. The coefficients of the Chebyshev series of $\arcsin x$ and $(\arcsin x)/x$ (App. C) are slowly descending. The infinite slope of $\arcsin x$ at $x = \pm 1$ renders both series inefficient, so we turn to $\frac{1}{x} \arcsin \frac{x}{\sqrt{2}}$ instead as configured in (D.10). Keeping terms up to f_{36} yields an estimated maximum absolute error of $\sum_{n=38}^N |f_n| \approx 1.4 \cdot 10^{-17}$ evaluated at $N = 108$. The relative error of the same polynomial is $\approx 1.9 \cdot 10^{-17}$. After four Newton iterations, this value drops only slightly to $1.8 \cdot 10^{-17}$; obviously, there is not much room to improve the polynomial representation w.r.t. an optimized *relative* error in cases where the amplitude of the function is small over the x -interval.

Example 5.7. The expansion for $\exp(x)$ in $-1 \leq x \leq 1$ reads [1, (9.6.19)] [28, (3.4.1e)][13, p. 69][9, (33)]

$$(5.13) \quad \exp(x) = 2 \sum_{n=0}^{\infty} I_n(1) T_n(x).$$

Truncation after $k = 14$ yields an estimated maximum absolute error of $\sum_{n=k+1}^N |f_n| \approx 4.9 \cdot 10^{-17}$ evaluated at $N = 42$. The relative error of the same polynomial is $\sum_{n=0}^N |\hat{a}_n| - 1 \approx 1.3 \cdot 10^{-16}$. After four Newton iterations, this value drops to $7.5 \cdot 10^{-17}$ with coefficients given in the following table, also listed as $f(x) \approx \sum_{n=0}^N d_n x^n$:

n	b_n	d_n
0	1.2660658777520083355982446	1.00000000000000002107745526254
1	1.1303182079849700544153921	1.000000000000000063548946139343
2	0.2714953395340765623657051	0.49999999999999997953936666685291
3	$0.4433684984866380495257150 \cdot 10^{-1}$	$0.166666666666666422610320391$
4	$0.5474240442093732650276168 \cdot 10^{-2}$	$0.41666666666669875817272051 \cdot 10^{-1}$
5	$0.5429263119139437503621352 \cdot 10^{-3}$	$0.8333333333602639662588442 \cdot 10^{-2}$
6	$0.4497732295429514665443872 \cdot 10^{-4}$	$0.1388888888702869286166025 \cdot 10^{-2}$
7	$0.3198436462401990501334121 \cdot 10^{-5}$	$0.1984126971086418099245159 \cdot 10^{-3}$
8	$0.1992124806672795001043316 \cdot 10^{-6}$	$0.2480158780231612103680909 \cdot 10^{-4}$
9	$0.1103677172551632915777862 \cdot 10^{-7}$	$0.2755735152373104259316644 \cdot 10^{-5}$
10	$0.5505896079551881657982078 \cdot 10^{-9}$	$0.2755725369287090362239172 \cdot 10^{-6}$
11	$0.2497956604792065959497342 \cdot 10^{-10}$	$0.2504783672757589754944252 \cdot 10^{-7}$
12	$0.1039151254481832513826561 \cdot 10^{-11}$	$0.2088034159586738951818317 \cdot 10^{-8}$
13	$0.3990676874210170341122722 \cdot 10^{-13}$	$0.1634581247676485771723867 \cdot 10^{-9}$
14	$0.1400237499722866786358850 \cdot 10^{-14}$	$0.1147074559772972471385170 \cdot 10^{-10}$

Example 5.8. The expansion for $J_0(\frac{\pi}{2}x)$ in $-1 \leq x \leq 1$ reads [15, 6.681.5]

$$(5.14) \quad J_0\left(\frac{\pi}{2}x\right) = 2 \sum'_{n=0,2,4,\dots} (-)^{n/2} J_{n/2}^2\left(\frac{\pi}{4}\right) T_n(x).$$

Truncation after $T_{16}(x)$ yields an estimated maximum absolute error of $\sum_{n=k+1}^N |f_n| \approx 7.3 \cdot 10^{-19}$ evaluated at $N = 48$. The relative error of the same polynomial is $\sum_{n=0}^N |\hat{a}_n| - 1 \approx 1.6 \cdot 10^{-18}$. After four Newton iterations, this value drops to $1.3 \cdot 10^{-18}$ with coefficients given in the following table:

n	b_n	d_n
0	0.7252769164405135618043045	0.999999999999999991311745
2	-0.2638108118461404734713153	-0.6168502750680847778603892
4	$0.1072184541022420669256084 \cdot 10^{-1}$	$0.9512606546288948620024320 \cdot 10^{-1}$
6	$-0.1885687642135952967199171 \cdot 10^{-3}$	$-0.6519837738512518004083602 \cdot 10^{-2}$
8	$0.1845983728936489887451460 \cdot 10^{-5}$	$0.2513602312234872245916252 \cdot 10^{-3}$
10	$-0.1150537142155094251800350 \cdot 10^{-7}$	$-0.6202064609606906421245435 \cdot 10^{-5}$
12	$0.4965029850154789447530764 \cdot 10^{-10}$	$0.1062698637612363296679714 \cdot 10^{-6}$
14	$-0.1571252252452718608949964 \cdot 10^{-11}$	$-0.1336990135568532922581048 \cdot 10^{-8}$
16	$0.3800986508122698831881511 \cdot 10^{-15}$	$0.1245507258981645953230933 \cdot 10^{-10}$

A set of b_j in

$$(5.15) \quad R(x) \equiv \frac{f(x)}{\sum_{j=0}^k b_j T_j(x)} - 1$$

found that way is also a starting point to calculate the solution with the minimax property of the relative error: This locates the local minima and maxima of $R(x)$, computes the mean of their absolute values, and iteratively adjusts the b_j such

that the absolute values of the new alternating extrema equal that mean. The corrections Δ_j to the b_j can be computed by expansion of (5.15) to first order in Δ_j keeping the abscissa of the extrema fixed, which ends up in a linear system of equations for the Δ_j .

Example 5.9. An IEEE “double” precision accuracy of $f(x) = \sin(\frac{\pi}{2}x)/x$ with a relative error smaller than $2^{-53} \approx 1.1 \cdot 10^{-16}$ needs $k = 16$. Truncation of the Chebyshev series of Example (4.3) for $f(x)$ after $k = 16$ yields an estimated maximum absolute error of $\sum_{n=k+1}^N |f_n| \approx 4.1 \cdot 10^{-19}$ evaluated at $N = 32$. The relative error of the same polynomial is $\sum_{n=0}^N |\hat{a}_n| - 1 \approx 3.8 \cdot 10^{-19}$. After four Newton iterations, this value drops to $3.5 \cdot 10^{-19}$ with coefficients b_n given in the following table:

n	b_n	d_n
0	1.276278962402265880207637	1.5707963267948966188688195
2	-0.2852615691810360095702941	-0.6459640975062461962319336
4	0.9118016006651802497767923 $\cdot 10^{-2}$	0.7969262624616554097627533 $\cdot 10^{-1}$
6	-0.1365875135419666724364765 $\cdot 10^{-3}$	-0.4681754135303468240882506 $\cdot 10^{-2}$
8	0.1184961857661690108288872 $\cdot 10^{-5}$	0.1604411847100114088031881 $\cdot 10^{-3}$
10	-0.6702791603827441081706121 $\cdot 10^{-8}$	-0.3598843013917326159520456 $\cdot 10^{-5}$
12	0.2667278599017903283863443 $\cdot 10^{-10}$	0.5692135656122429901944357 $\cdot 10^{-7}$
14	-0.7872922004615709018594325 $\cdot 10^{-13}$	-0.6684369436484103757933363 $\cdot 10^{-9}$
16	0.1791929094718284072119916 $\cdot 10^{-15}$	0.5871793257572873247522307 $\cdot 10^{-11}$

The actual relative error of this approximation is shown in Fig. 1 as a continuous line, with a maximum of $2.9 \cdot 10^{-19}$. The dashed line with a relative error of $2.6 \cdot 10^{-19}$ in comparison results from further minimax optimization with coefficients shown in the next table:

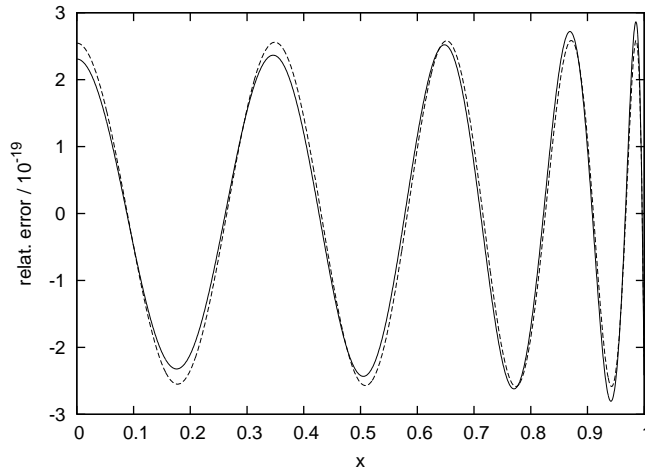


FIGURE 1. The relative error $R(x)$ to $f(x) = \sin(\pi x/2)/x$ for both tabulated parameter sets of $\sum_{n=0,2,\dots}^{16} b_n T_n(x)$ of Example 5.9.

n	b_n	d_n
0	1.2762789624022658802075437	1.5707963267948966188314659
2	-0.2852615691810360095705230	- 0.6459640975062461915471363
4	0.9118016006651802497528156 $\cdot 10^{-2}$	0.7969262624616544421893744 $\cdot 10^{-1}$
6	-0.1365875135419666726405733 $\cdot 10^{-3}$	-0.4681754135302704719117724 $\cdot 10^{-2}$
8	0.1184961857661689920542732 $\cdot 10^{-5}$	0.1604411847070460989830944 $\cdot 10^{-3}$
10	-0.6702791603827612608171959 $\cdot 10^{-8}$	-0.3598843007652658555526627 $\cdot 10^{-5}$
12	0.2667278599019855592489579 $\cdot 10^{-10}$	0.5692134921914455833455723 $\cdot 10^{-7}$
14	-0.7872921659616258733890169 $\cdot 10^{-13}$	-0.6684324580312975131354658 $\cdot 10^{-9}$
16	0.1791589025538146793760922 $\cdot 10^{-15}$	0.5870678918883399413795788 $\cdot 10^{-11}$

Example 5.10. As an example for (4.18), consider $\exp(x) = \sum_{n=0}'^{\infty} f_n T_n^*(x)$ over $0 \leq x \leq 1$ [1, (4.2.48)][5, 19]. The f_n are represented via [1, (9.6.26)] through modified Bessel Functions I_n ,

$$(5.16) \quad f_n = 2\sqrt{e}I_n(1/2); \quad f_{n+1} = -4nf_n + f_{n-1},$$

n	f_n
0	3.506775308754180791443893
1	0.8503916537808109665352350
2	0.1052086936309369253029528
3	0.008722104733315564111612874
4	0.0005434368311501559635982758
5	0.00002711543491306869404046064

Truncation of the Chebyshev series for $f(x)$ after $k = 3$ yields an estimated maximum absolute error of $\sum_{n=k+1}^N |f_n| \approx 5.7 \cdot 10^{-4}$ evaluated at $N = 9$. The relative error of the same polynomial is $\sum_{n=0}^N |\hat{a}_n| - 1 \approx 5.1 \cdot 10^{-4}$. After four Newton iterations, this value drops to $4.0 \cdot 10^{-4}$ with coefficients given in the following table:

n	b_n
0	1.753387654377090395721946
1	0.8503902561425088936327743
2	0.1051918520893768747555014
3	0.008587089960927766771654559

If we proceed to $k = 12$ at $N = 36$, the estimated maximum relative error becomes $6.1 \cdot 10^{-18}$ with the following coefficients:

n	b_n	d_n
0	1.7533876543770903957219464	1.000000000000000060373678
1	0.8503916537808109665352350	0.99999999999999978889799411
2	0.1052086936309369253029528	0.5000000000001216148194572
3	$0.8722104733315564111612874 \cdot 10^{-2}$	0.1666666666639271874501180
4	$0.5434368311501559635982758 \cdot 10^{-3}$	$0.4166666669859109153386033 \cdot 10^{-1}$
5	$0.2711543491306869404045765 \cdot 10^{-4}$	$0.8333333112815145481691497 \cdot 10^{-2}$
6	$0.1128132888782082788967416 \cdot 10^{-5}$	$0.1388889862738933258163839 \cdot 10^{-2}$
7	$0.4024558229870710027066467 \cdot 10^{-7}$	$0.1984098287973665146421103 \cdot 10^{-3}$
8	$0.1256584418283842256517024 \cdot 10^{-8}$	$0.2480734627092463176804164 \cdot 10^{-4}$
9	$0.3488091362080888722258141 \cdot 10^{-10}$	$0.2747848541489261879291146 \cdot 10^{-5}$
10	$0.8715278679388174731063544 \cdot 10^{-12}$	$0.2827881515524984459349078 \cdot 10^{-6}$
11	$0.1979783472020383084286900 \cdot 10^{-13}$	$0.2086709669366350082217004 \cdot 10^{-7}$
12	$0.4103178180353125619414324 \cdot 10^{-15}$	$0.3441995330913567239602395 \cdot 10^{-8}$

Equilibration of the local extrema with the following coefficients reduces this error to $5.0 \cdot 10^{-18}$:

n	b_n	d_n
0	1.7533876543770903961757996	1.000000000000000049913878
1	0.8503916537808109674449984	0.99999999999999982006556032
2	0.1052086936309369262175803	0.5000000000001063630793784
3	$0.8722104733315565035195027 \cdot 10^{-2}$	0.1666666666642173677701902
4	$0.5434368311501568988037582 \cdot 10^{-3}$	$0.4166666669575619866774579 \cdot 10^{-1}$
5	$0.2711543491306964588901369 \cdot 10^{-4}$	$0.8333333129071360606767972 \cdot 10^{-2}$
6	$0.1128132888783054546376918 \cdot 10^{-5}$	$0.1388889803905621871712292 \cdot 10^{-2}$
7	$0.4024558229970401854905218 \cdot 10^{-7}$	$0.1984099684263107292542263 \cdot 10^{-3}$
8	$0.1256584419307581158766302 \cdot 10^{-8}$	$0.2480712594971168345247335 \cdot 10^{-4}$
9	$0.3488091466142981067488685 \cdot 10^{-10}$	$0.2748077480706561519632930 \cdot 10^{-5}$
10	$0.8715288225355426665019433 \cdot 10^{-12}$	$0.2826376902452534679601836 \cdot 10^{-6}$
11	$0.1979820112783685973416909 \cdot 10^{-13}$	$0.2092376435267840500466468 \cdot 10^{-7}$
12	$0.4092071997914099904014169 \cdot 10^{-15}$	$0.3432678789827820176761249 \cdot 10^{-8}$

6. SUMMARY

Besides some generic algorithms to compute the Chebyshev series of inverse polynomials, there are two specific aspects that facilitate this task: (i) the expansion coefficients can be derived from the partial fractions of the inverse polynomial. (ii) Expansion coefficients with indices larger than the polynomial degree are recursively linked to those of lower order. (iii) An algorithm has been presented which derives a polynomial of a given degree such that the first terms of the Chebyshev expansion of the relative error of a given function represented by this polynomial vanish.

APPENDIX A. CHEBYSHEV SERIES OF $\ln(1+x)$

The integral representation

$$(A.1) \quad \ln(1+x) = \int \frac{dx}{1+x}$$

and term-by-term integration of (2.10) on the r.h.s. with (2.9) yield that the Chebyshev coefficients of

$$(A.2) \quad f(x) = \ln(1+x) \equiv \sum_{n=0}^{\infty} f_n T_n^*(x) \quad 0 \leq x \leq 1,$$

obey

$$(A.3) \quad 2nf_n = a_{n+1,1}(-3) - a_{n-1,1}(-3), \quad n \geq 1,$$

explicitly [13, p. 88][14]

$$(A.4) \quad f_n = \frac{2(-)^{n+1}}{n(3+2\sqrt{2})^n}, \quad n \geq 1$$

from (2.11), as tabulated in [1, 4.1.45]. The missing f_0 is

$$(A.5) \quad f_0 = \frac{2}{\pi} \int_0^1 \frac{\ln(1+x)}{\sqrt{x(1-x)}} dx = 2 \ln \frac{3+2\sqrt{2}}{4},$$

because insertion of $x = 1$ in (A.2) yields

$$(A.6) \quad f_0 = 2 \left(f(1) - \sum_{n=1}^{\infty} f_n \right)$$

and $f(1) = \ln 2$ and $\sum_{n=1}^{\infty} f_n = 2 \ln(1 + \frac{1}{3+2\sqrt{2}})$ via [1, 4.1.24].

APPENDIX B. CHEBYSHEV SERIES OF $\arctan x$

Integrating (2.7) over x with

$$(B.1) \quad \int \frac{1}{1+x^2} dx = \arctan x$$

and (1.5) we get [13, p. 89][14]

$$(B.2) \quad \arctan x = 2 \sum_{j=1,3,5,7,\dots} \frac{(-)^{\lfloor j/2 \rfloor}}{j(1+\sqrt{2})^j} T_j(x), \quad -1 \leq x \leq 1,$$

in particular at $x = 1$

$$(B.3) \quad \frac{\pi}{8} = \sum_{j=1,3,5,7,\dots} \frac{(-)^{\lfloor j/2 \rfloor}}{j(1+\sqrt{2})^j}.$$

From (B.2) and (1.15), the coefficients of

$$(B.4) \quad \frac{\arctan x}{x} \equiv \sum_{n=0,2,4,6,\dots}' g_n T_n(x)$$

as listed in [1, (4.4.50)][5] follow recursively, where $g_0 = 2 \ln(1 + \sqrt{2})$ is obtained via [15, 4.531.12].

APPENDIX C. CHEBYSHEV SERIES OF $\arcsin x$

The series of $\arcsin x = \sum_{n=1,3,5,\dots}^{\infty} g_n T_n(x)$ starts with $g_1 = 4/\pi$. A combination of [1, (4.4.58)], (1.4), (1.5) and [28, (3.4.1d)]

$$(C.1) \quad \sqrt{1-x^2} = \frac{4}{\pi} \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \frac{1}{1-n^2} T_n(x)$$

yields

$$(C.2) \quad (n+1)g_{n+1} + (n-1)g_{n-1} = \frac{8}{\pi} \frac{n}{n^2-1}, \quad n = 2, 4, 6, \dots$$

in this case, which can be unwound as $g_n = 4/(\pi n^2)$. To find a formulation with controlled *relative* error, we would switch to $h(x) = (\arcsin x)/x = \sum_{n=0}^{\infty} h_n T_n(x)$ to remove the zero in the spirit of example 5.5. With (1.15), the expansion coefficients are

$$(C.3) \quad h_0 = 8\beta(2)/\pi,$$

$$(C.4) \quad h_{n+2} = -h_n + \frac{8}{\pi(n+1)^2},$$

where $\beta(2) \approx 0.915965594177219015054603515$ is Catalan's constant [1, Tab 23.3][15, 0.234.3].

APPENDIX D. CHEBYSHEV SERIES OF $\arcsin(x/\sqrt{2})$

The coefficients of

$$(D.1) \quad \arcsin(x/\sqrt{2}) = \sum_{n=1,3,5,\dots}^{\infty} k_n T_n(x)$$

are found by partial integration of $\int \arcsin([\cos \theta]/\sqrt{2}) \cos(n\theta) d\theta$

$$(D.2) \quad k_n = \frac{1}{n\pi} \left[\int_0^\pi \frac{\cos[(n-1)\theta]}{\sqrt{2-\cos^2 \theta}} d\theta - \int_0^\pi \frac{\cos[(n+1)\theta]}{\sqrt{2-\cos^2 \theta}} d\theta \right], \quad n \text{ odd}$$

where

$$(D.3) \quad \int_0^\pi \frac{\cos(2m\theta)}{\sqrt{2-\cos^2 \theta}} d\theta = G_{2m} = \begin{cases} \sqrt{2}F(\frac{1}{\sqrt{2}}) \approx 2.6220575542921198104648395899, & m = 0, \\ \sqrt{2}[3F(\frac{1}{\sqrt{2}}) - 4E(\frac{1}{\sqrt{2}})] \approx 0.22577708482093539558499460534, & m = 1, \end{cases}$$

are Complete Elliptic Integrals. To find a recurrence for these

$$(D.4) \quad G_s \equiv \int_{-1}^1 \frac{T_s(x)}{\sqrt{(2-x^2)(1-x^2)}} dx,$$

we apply the method of [1, (17.1.4)] to the quartic $y^2 \equiv (2-x^2)(1-x^2)$, with $d(yT_s(x))/dx = y(dT_s(x)/dx) + T_s \frac{1}{2y}(T_3(x) - 3T_1(x))$, insert (1.6) for the derivative on the r.h.s, replace the first y on the r.h.s. by $y^2/y = (T_4/8 - T_2 + 7/8)/y$, expand all products with (1.4), and finally insert the upper limit $x = 1$ where $y(x)T_s(x) = 0$:

$$(D.5) \quad G_{s+3} + G_{|s-3|} - 3(G_{s+1} + G_{|s-1|}) + \frac{s}{2} \sum_{\substack{l=0 \\ l-s \text{ odd}}}^{s-1} [G_{l+4} + G_{|l-4|} - 8(G_{l+2} + G_{|l-2|}) + 14G_l] = 0.$$

Inserting $s = 1, 3, 5$ and 7 , for example, yields

$$(D.6) \quad 3G_4 - 12G_2 + G_0 = 0,$$

$$(D.7) \quad 5G_6 - 27G_4 + 15G_2 - G_0 = 0,$$

$$(D.8) \quad 7G_8 - 41G_6 + 29G_4 - 3G_2 = 0,$$

$$(D.9) \quad 9G_{10} - 55G_8 + 43G_6 - 5G_4 = 0,$$

and generates k_1 to k_9 in (D.2) from G_0 and G_2 shown in (D.3). A slowly converging series expansion is also known [4, 806.01]. With (1.15) we find the coefficients $f_n = 2k_{n-1} - f_{n-2}$ for

$$(D.10) \quad \frac{1}{x} \arcsin \frac{x}{\sqrt{2}} = \sum'_{n=0,2,4,\dots}^{\infty} f_n T_n(x), \quad -1 \leq x \leq 1,$$

starting at

$$f_0 = \frac{4}{\pi} \int_0^1 \frac{\arcsin \frac{x}{\sqrt{2}}}{x\sqrt{1-x^2}} dx = \sqrt{2} \sum_{l=0}^{\infty} \frac{[(2l-1)!!]^2}{2^l(2l+1)[(2l)!!]^2}$$

$$(D.11) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{q=1}^{\infty} \frac{1}{4q-1} [G_{4q-2} - G_{4q}] \approx 1.4866664932871034689603296833.$$

The four coefficients α_i that span

(D.12)

$$k_{n-1} = \frac{\sqrt{2}}{\pi} \left[\alpha_1 K\left(\frac{1}{\sqrt{2}}\right) + \alpha_2 E\left(\frac{1}{\sqrt{2}}\right) \right], \quad f_n = \frac{\sqrt{2}}{\pi} \left[\alpha_3 K\left(\frac{1}{\sqrt{2}}\right) + \alpha_4 E\left(\frac{1}{\sqrt{2}}\right) \right] + (-)^{[n/2]} f_0,$$

start as follows:

n	α_1	α_2	α_3	α_4
2	-2	4	-4	8
4	-26/9	4	-16/9	0
6	-638/75	292/25	-3428/225	584/25
8	-22702/735	212/5	-513088/11025	1536/25
10	-23722/189	4652/27	-6763436/33075	191128/675
12	-463174/847	2252/3	-3558618544/4002075	822272/675
14	-162508858/65065	8691484/2535	-2777152623884/676350675	643269592/114075

Because $T_{2j}(x) = T_j^*(x^2)$, the following numbers coincide with [1, (4.4.51)] up

to a factor $\sqrt{2}$:

n	f_n	n	f_n
2	0.3885303371652290716432228 $\cdot 10^{-1}$	6	0.2884218334475536563483289 $\cdot 10^{-3}$
4	0.2885441422084471126676825 $\cdot 10^{-2}$	10	0.4158477878052832866177270 $\cdot 10^{-5}$
8	0.3322367192785279209254231 $\cdot 10^{-4}$	14	0.7550078449371525934251585 $\cdot 10^{-7}$
12	0.5496504525974164467345493 $\cdot 10^{-6}$	18	0.1542180379281470021561106 $\cdot 10^{-8}$
16	0.1067193805629843129424091 $\cdot 10^{-7}$	22	0.3383885639342775871004709 $\cdot 10^{-10}$
20	0.2268114598545151963877153 $\cdot 10^{-9}$	26	0.7791139213632464421446539 $\cdot 10^{-12}$
24	0.5108937524377197224216916 $\cdot 10^{-11}$	30	0.1856972621821342234640637 $\cdot 10^{-13}$
28	0.1198378589352895337866326 $\cdot 10^{-12}$	34	0.4542792886328823081478511 $\cdot 10^{-15}$
32	0.2896189154386304361020997 $\cdot 10^{-14}$	38	0.1134144256904559996509711 $\cdot 10^{-16}$
36	0.7161678029265506176831289 $\cdot 10^{-16}$		

APPENDIX E. CHEBYSHEV SERIES OF $\psi(x+2)$

An expansion of the Digamma function [34] is [1, (6.3.16)]

$$(E.1) \quad \psi(2+x) = 1 - \gamma + x \sum_{k=2}^{\infty} \frac{1}{k(x+k)},$$

where $\gamma \approx 0.5772$ is Euler's constant. Employing $a_{n,1}(-k)$ of (2.4),

$$(E.2) \quad \frac{1}{x+k} = -\frac{1}{-k-x} = \frac{2}{\sqrt{k^2-1}} \sum_{n=0}^{\infty} \frac{(-)^n}{(k+\sqrt{k^2-1})^n} T_n(x), \quad -1 \leq x \leq 1,$$

$$(E.3) \quad \psi(2+x) = 1 - \gamma + 2x \sum_{k=2}^{\infty} \frac{1}{k\sqrt{k^2-1}} \sum_{n=0}^{\infty} \frac{(-)^n}{(k+\sqrt{k^2-1})^n} T_n(x).$$

The auxiliary definition

$$(E.4) \quad K_n \equiv \sum_{k=2}^{\infty} \frac{1}{k\sqrt{k^2-1}(k+\sqrt{k^2-1})^n}, \quad n = 0, 1, 2, \dots$$

turns (E.3) with the aid of (1.4) into

$$(E.5) \quad \psi(x+2) = (1-\gamma-K_1)T_0(x) - \sum_{n=1}^{\infty} (-)^n (K_{n-1}+K_{n+1})T_n(x), \quad -1 \leq x \leq 1,$$

where

$$(E.6) \quad K_n + K_{n+2} = 2 \sum_{k=2}^{\infty} \frac{1}{\sqrt{k^2-1}(k+\sqrt{k^2-1})^{n+1}}, \quad n = 0, 1, 2, \dots$$

Alternatives to the slowly converging original series (E.4) at small n are obtained in terms of the Riemann Zeta function ζ after reducing the fraction in (E.4) and/or (E.6) by $k - \sqrt{k^2-1}$,

$$(E.7) \quad K_0 = \sum_{k=2}^{\infty} \frac{1}{k^2} \left(1 - \frac{1}{k^2}\right)^{-1/2} = \sum_{l=0}^{\infty} (-)^l \binom{-1/2}{l} [\zeta(2l+2) - 1],$$

$$(E.8) \quad K_1 = \sum_{k=2}^{\infty} \frac{k - \sqrt{k^2-1}}{k\sqrt{k^2-1}} = \sum_{l=1}^{\infty} (-)^l \binom{-1/2}{l} [\zeta(2l+1) - 1],$$

$$(E.9) \quad K_0 + K_2 = 2 \sum_{l=1}^{\infty} (-)^l \binom{-1/2}{l} [\zeta(2l) - 1],$$

$$(E.10) \quad K_1 + K_3 = 2 \sum_{l=2}^{\infty} (-)^l \left(\binom{-1/2}{l} + \binom{1/2}{l} \right) [\zeta(2l-1) - 1],$$

$$(E.11) \quad K_n + K_{n+2} = 2 \sum_{l=[(n+3)/2]}^{\infty} (-)^l [\zeta(2l-n) - 1] \sum_{s=0}^{n+1} \binom{n+1}{s} \binom{(s-1)/2}{l}.$$

n	K_n	n	K_n
0	0.6942240199692270653811973	1	0.1181923495113155830503315
2	$0.2615575442260127035429158 \cdot 10^{-1}$	3	$0.6357242927298094244957032 \cdot 10^{-2}$
4	$0.1613702909326556648518537 \cdot 10^{-2}$	5	$0.4189942166841513997803225 \cdot 10^{-3}$
6	$0.1101726048982138724638504 \cdot 10^{-3}$	7	$0.2918277395837793537278094 \cdot 10^{-4}$
8	$0.7763995103341854698876680 \cdot 10^{-5}$	9	$0.2071120322602199079344235 \cdot 10^{-5}$
10	$0.5534045978754736410165904 \cdot 10^{-6}$	11	$0.1480224417758054637706871 \cdot 10^{-6}$
12	$0.3961806941781982189370558 \cdot 10^{-7}$	13	$0.1060807013890109056491206 \cdot 10^{-7}$
14	$0.2841134565373781348071928 \cdot 10^{-8}$	15	$0.7610594780500236739360477 \cdot 10^{-9}$
16	$0.2038876075855356359426642 \cdot 10^{-9}$	17	$0.5462507275750785409310247 \cdot 10^{-10}$
18	$0.1463563991421891531670298 \cdot 10^{-10}$	19	$0.3921418684181661587649434 \cdot 10^{-11}$
20	$0.1050708536652289553110610 \cdot 10^{-11}$	21	$0.2815309431326615497301838 \cdot 10^{-12}$
22	$0.7543503527420115661214215 \cdot 10^{-13}$	23	$0.2021259323594124792356794 \cdot 10^{-13}$
24	$0.5415919982188370035264436 \cdot 10^{-14}$	25	$0.1451186573479985220441655 \cdot 10^{-14}$
26	$0.3888434449455691142432431 \cdot 10^{-15}$	27	$0.1041901454404881356692972 \cdot 10^{-15}$
28	$0.2791764103483896329393674 \cdot 10^{-16}$		

As a by-product, insertion of $x = \pm 1$ in (E.3) with $\psi(1) = -\gamma$ shows

$$(E.12) \quad \sum_{k=2}^{\infty} \frac{1}{k\sqrt{k^2-1}} \frac{k-1+\sqrt{k^2-1}}{k+1+\sqrt{k^2-1}} = \frac{1}{2},$$

$$(E.13) \quad \sum_{k=2}^{\infty} \frac{1}{k\sqrt{k^2-1}} \frac{k+1+\sqrt{k^2-1}}{k-1+\sqrt{k^2-1}} = 1.$$

Linear combinations of these two equations are

$$(E.14) \quad \sum_{k=2}^{\infty} \frac{(k-3)(k+\sqrt{k^2-1})}{k\sqrt{k^2-1}[(k+\sqrt{k^2-1})^2-1]} = 0,$$

$$(E.15) \quad \sum_{k=2}^{\infty} \frac{k+\sqrt{k^2-1}}{k\sqrt{k^2-1}[(k+\sqrt{k^2-1})^2-1]} = \frac{1}{8},$$

and these two can be combined to

$$(E.16) \quad \sum_{k=2}^{\infty} \frac{k+\sqrt{k^2-1}}{\sqrt{k^2-1}[(k+\sqrt{k^2-1})^2-1]} = \frac{3}{8}.$$

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